

LOCAL RAMSEY THEORY. AN ABSTRACT APPROACH

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ABSTRACT. Given a topological Ramsey space (\mathcal{R}, \leq, r) , we extend the notion of semiselective coideal to sets $\mathcal{H} \subseteq \mathcal{R}$ and study conditions for \mathcal{H} that will enable us to make the structure $(\mathcal{R}, \mathcal{H}, \leq, r)$ a Ramsey space (not necessarily topological) and also study forcing notions related to \mathcal{H} which will satisfy abstract versions of interesting properties of the corresponding forcing notions in the realm of Ellentuck's space (see [7, 15]). This extends results from [8, 17] to the most general context of topological Ramsey spaces. As applications, we prove that for every topological Ramsey space \mathcal{R} , under suitable large cardinal hypotheses every semiselective ultrafilter $\mathcal{U} \subseteq \mathcal{R}$ is generic over $L(\mathbb{R})$; and that given a semiselective coideal $\mathcal{H} \subseteq \mathcal{R}$, every definable subset of \mathcal{R} is \mathcal{H} -Ramsey. This generalizes the corresponding results for the case when \mathcal{R} is equal to Ellentuck's space (see [8, 3]).

1. INTRODUCTION

Let $A \subseteq \mathbb{N}$ be given and consider the set $A^{[\infty]} = \{X \subset A : |X| = \infty\}$. Consider the sets of the form $[a, A] = \{B \in \mathbb{N}^{[\infty]} : a \sqsubset B \subseteq A\}$, where a is a finite $\mathbb{N}^{[\infty]}$, $A \in \mathbb{N}^{[\infty]}$ and $a \sqsubset B$ means that a is an initial segment of B . For a family $\mathcal{H} \subseteq \mathbb{N}^{[\infty]}$, a set $\mathcal{X} \subseteq \mathbb{N}^{[\infty]}$ is said to be \mathcal{H} -Ramsey if for every nonempty $[a, A]$ with $A \in \mathcal{H}$ there exists $B \in \mathcal{H} \cap A^{[\infty]}$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \cap \mathcal{X} = \emptyset$. \mathcal{X} is said to be \mathcal{H} -Ramsey null if for every nonempty $[a, A]$ with $A \in \mathcal{H}$ there exists $B \in \mathcal{H} \cap [a, A]$ such that $[a, B] \cap \mathcal{X} = \emptyset$.

Local Ramsey theory includes the study and characterization of the property defined above, which is a relativized version of the completely Ramsey property (see [12, 7]). In [15], Mathias introduces the happy families (or selective coideals) of subsets of \mathbb{N} and relativizes the notion of completely Ramsey subsets of $\mathbb{N}^{[\infty]}$ to such families. Then he proves that analytic sets are \mathcal{U} -Ramsey when \mathcal{U} is a Ramsey ultrafilter and generalizes this result for arbitrary happy families. In [8], Farah gives an answer to the question of Todorcevic: what are the combinatorial properties of the family \mathcal{H} of ground model subsets of \mathbb{N} which warranties diagonalization of the Borel partitions? This is done by imposing a condition on \mathcal{H} which is weaker than selectivity: the notion of semiselectivity. Farah shows not only that the semiselectivity of \mathcal{H} is enough to make the \mathcal{H} -Ramsey property equivalent to the abstract Baire property with respect to \mathcal{H} , but also shows that this latter equivalence characterizes semiselectivity. In [17], a step toward the understanding of the local Ramsey property within the most general context of topological Ramsey spaces is done.

In this work, given a topological Ramsey space (\mathcal{R}, \leq, r) , we extend the notion of semiselective coideal to sets $\mathcal{H} \subseteq \mathcal{R}$ and study conditions for \mathcal{H} that will enable us to make the structure $(\mathcal{R}, \mathcal{H}, \leq, r)$ a Ramsey space (not necessarily topological) and also study forcing notions related to \mathcal{H} which will satisfy abstract versions of interesting properties of the corresponding forcing notions in the realm of Ellentuck's space, extending results from [8, 17] to the most general context of topological Ramsey spaces. As applications, we prove that for

every topological Ramsey space \mathcal{R} , under suitable large cardinal hypotheses every semiselective ultrafilter $\mathcal{U} \subseteq \mathcal{R}$ is generic over $L(\mathbb{R})$; and that given a semiselective coideal $\mathcal{H} \subseteq \mathcal{R}$, every definable subset of \mathcal{R} is \mathcal{H} -Ramsey. This generalizes the corresponding results for the case when \mathcal{R} is equal to Ellentuck's space (see [8, 3]).

The structure of this work is as follows: Section 2 is a short introduction to the theory of topological Ramsey spaces. In Section 3, we extend the notion of semiselective coideal to subsets \mathcal{H} of a topological Ramsey space \mathcal{R} and study conditions for \mathcal{H} that will enable us to make the structure $(\mathcal{R}, \mathcal{H}, \leq, r)$ a Ramsey space. In particular, in this section we study the characterization of the corresponding abstract version of the \mathcal{H} -Ramsey property. In section 4, it is shown that the family of \mathcal{H} -Ramsey subsets of \mathcal{R} is closed under the Souslin operation, if \mathcal{H} is semiselective. In Section 5 we introduce an abstract version of selective coideal. This is then connected with Section 6 where we study forcing notions related to semiselectivity as defined in Section 3. Finally, in Section 7 we apply the main results in the previous Sections to prove that under suitable large cardinal hypotheses every semiselective ultrafilter $\mathcal{U} \subseteq \mathcal{R}$ is generic over $L(\mathbb{R})$ and that given a semiselective coideal $\mathcal{H} \subseteq \mathcal{R}$, every definable subset of \mathcal{R} is \mathcal{H} -Ramsey.

2. TOPOLOGICAL RAMSEY SPACES

The definitions and results throughout this section are taken from [24]. A previous presentation can be found in [2].

2.1. Metrically closed spaces and approximations. Consider a triplet of the form (\mathcal{R}, \leq, r) , where \mathcal{R} is a set, \leq is a quasi order on \mathcal{R} and $r : \mathbb{N} \times \mathcal{R} \rightarrow \mathcal{AR}$ is a function with range \mathcal{AR} . For every $n \in \mathbb{N}$ and every $A \in \mathcal{R}$, let us write

$$(1) \quad r_n(A) := r(n, A)$$

We say that $r_n(A)$ is **the n th approximation of A** . We will reserve capital letters $A, B \dots$ for elements in \mathcal{R} while lowercase letters $a, b \dots$ will denote elements of \mathcal{AR} . In order to capture the combinatorial structure required to ensure the provability of an Ellentuck type Theorem, some assumptions on (\mathcal{R}, \leq, r) will be imposed. The first is the following:

(A.1) [Metatrization]

- (A.1.1) For any $A \in \mathcal{R}$, $r_0(A) = \emptyset$.
- (A.1.2) For any $A, B \in \mathcal{R}$, if $A \neq B$ then $(\exists n) (r_n(A) \neq r_n(B))$.
- (A.1.3) If $r_n(A) = r_m(B)$ then $n = m$ and $(\forall i < n) (r_i(A) = r_i(B))$.

Take the discrete topology on \mathcal{AR} and endow $\mathcal{AR}^{\mathbb{N}}$ with the product topology; this is the metric space of all the sequences of elements of \mathcal{AR} . The set \mathcal{R} can be identified with the corresponding image in $\mathcal{AR}^{\mathbb{N}}$. We will say that \mathcal{R} is **metrically closed** if, as a subspace $\mathcal{AR}^{\mathbb{N}}$ with the inherited topology, it is closed. The basic open sets generating the metric topology on \mathcal{R} inherited from the product topology of $\mathcal{AR}^{\mathbb{N}}$ are of the form:

$$(2) \quad [a] = \{B \in \mathcal{R} : (\exists n)(a = r_n(B))\}$$

where $a \in \mathcal{AR}$. Let us define the **length** of a , as the unique integer $|a| = n$ such that $a = r_n(A)$ for some $A \in \mathcal{R}$. For every $n \in \mathbb{N}$, let

$$(3) \quad \mathcal{AR}_n := \{a \in \mathcal{AR} : |a| = n\}$$

Hence,

$$(4) \quad \mathcal{AR} = \bigcup_{n \in \mathbb{N}} \mathcal{AR}_n$$

The **Ellentuck type neighborhoods** are of the form:

$$(5) \quad [a, A] = \{B \in [a] : B \leq A\} = \{B \in \mathcal{R} : (\exists n) a = r_n(B) \text{ \& } B \leq A\}$$

where $a \in \mathcal{AR}$ and $A \in \mathcal{R}$.

We will use the symbol $[n, A]$ to abbreviate $[r_n(A), A]$.

Let

$$(6) \quad \mathcal{AR} \restriction A = \{a \in \mathcal{AR} : [a, A] \neq \emptyset\}$$

Given a neighborhood $[a, A]$ and $n \geq |a|$, let $r_n[a, A]$ be the image of $[a, A]$ by the function r_n , i.e.,

$$(7) \quad r_n[a, A] = \{r_n(B) : B \in [a, A]\}$$

2.2. Ramsey sets. A set $\mathcal{X} \subseteq \mathcal{R}$ is **Ramsey** if for every neighborhood $[a, A] \neq \emptyset$ there exists $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \cap \mathcal{X} = \emptyset$. A set $\mathcal{X} \subseteq \mathcal{R}$ is **Ramsey null** if for every neighborhood $[a, A]$ there exists $B \in [a, A]$ such that $[a, B] \cap \mathcal{X} = \emptyset$.

2.3. Topological Ramsey spaces. We say that (\mathcal{R}, \leq, r) is a **topological Ramsey space** iff subsets of \mathcal{R} with the Baire property are Ramsey and meager subsets of \mathcal{R} are Ramsey null.

Given $a, b \in \mathcal{AR}$, write

$$(8) \quad a \sqsubseteq b \text{ iff } (\exists A \in \mathcal{R}) (\exists m, n \in \mathbb{N}) m \leq n, a = r_m(A) \text{ and } b = r_n(A).$$

By A.1, \sqsubseteq can be proven to be a partial order on \mathcal{AR} .

(A.2) [Finitization] There is a quasi order \leq_{fin} on \mathcal{AR} such that:

$$(A.2.1) \quad A \leq B \text{ iff } (\forall n) (\exists m) (r_n(A) \leq_{fin} r_m(B)).$$

$$(A.2.2) \quad \{b \in \mathcal{AR} : b \leq_{fin} a\} \text{ is finite, for every } a \in \mathcal{AR}.$$

$$(A.2.3) \quad \text{If } a \leq_{fin} b \text{ and } c \sqsubseteq a \text{ then there is } d \sqsubseteq b \text{ such that } c \leq_{fin} d.$$

Given $A \in \mathcal{R}$ and $a \in \mathcal{AR} \restriction A$, we define the **depth** of a in A as

$$(9) \quad \text{depth}_A(a) := \min\{n : a \leq_{fin} r_n(A)\}$$

(A.3) [Amalgamation] Given a and A with $\text{depth}_A(a) = n$, the following holds:

$$(A.3.1) \quad (\forall B \in [n, A]) ([a, B] \neq \emptyset).$$

$$(A.3.2) \quad (\forall B \in [a, A]) (\exists A' \in [n, A]) ([a, A'] \subseteq [a, B]).$$

(A.4) [Pigeonhole Principle] Given a and A with $\text{depth}_A(a) = n$, for every $\mathcal{O} \subseteq \mathcal{AR}_{|a|+1}$ there is $B \in [n, A]$ such that $r_{|a|+1}[a, B] \subseteq \mathcal{O}$ or $r_{|a|+1}[a, B] \subseteq \mathcal{O}^c$.

Theorem 2.1 (Todorćevic, [24]). **[Abstract Ellentuck Theorem]** Any (\mathcal{R}, \leq, r) with \mathcal{R} metrically closed and satisfying (A.1)-(A.4) is a topological Ramsey space.

Besides [24], we refer the reader to [4, 5, 6, 17, 18, 19, 20, 21, 25] for further developments on the theory of Ramsey spaces.

3. ABSTRACT SEMISELECTIVITY

Notation. Given a triple (\mathcal{R}, \leq, r) as defined in the previous section and $\mathcal{H} \subseteq \mathcal{R}$, let $\mathcal{H} \restriction A = \{B \in \mathcal{H} : B \leq A\}$.

Definition 3.1. Consider a triple (\mathcal{R}, \leq, r) satisfying **A1** – **A4**. Given $\mathcal{H} \subseteq \mathcal{R}$, we say that \mathcal{H} is a **coideal** if it satisfies the following:

- (a) For all $A, B \in \mathcal{R}$, if $A \in \mathcal{H}$ and $A \leq B$ then $B \in \mathcal{H}$.
- (b) (**A3** mod \mathcal{H}) For all $A \in \mathcal{H}$ and $a \in \mathcal{AR} \restriction A$, the following holds:
 - $[a, B] \neq \emptyset$ for all $B \in [\text{depth}_A(a), A] \cap \mathcal{H}$.
 - If $B \in \mathcal{H} \restriction A$ and $[a, B] \neq \emptyset$ then there exists $A' \in [\text{depth}_A(a), A] \cap \mathcal{H}$ such that $\emptyset \neq [a, A'] \subseteq [a, B]$.
- (c) (**A4** mod \mathcal{H}) Let $A \in \mathcal{H}$ and $a \in \mathcal{AR} \restriction A$ be given. For all $\mathcal{O} \subseteq \mathcal{AR}_{|a|+1}$ there exists $B \in [\text{depth}_A(a), A] \cap \mathcal{H}$ such that $r_{|a|+1}[a, B] \subseteq \mathcal{O}$ or $r_{|a|+1}[a, B] \cap \mathcal{O} = \emptyset$.

We will study conditions for a coideal $\mathcal{H} \subseteq \mathcal{R}$ such that the structure $(\mathcal{R}, \mathcal{H}, \leq, \leq, r, r)$ is a Ramsey space (in the sense of [24], Chapter 4). For short, from now on we will write $(\mathcal{R}, \mathcal{H}, \leq, r)$ instead of $(\mathcal{R}, \mathcal{H}, \leq, \leq, r, r)$. It is easy to see that $(\mathcal{R}, \mathcal{H}, \leq, r)$ satisfies **A1** – **A4** for general Ramsey spaces. Therefore, we know from the Abstract Ramsey Theorem (Theorem 4.27 in [24]) that if \mathcal{H} is closed in $(\mathcal{AR})^\mathbb{N}$ then $(\mathcal{R}, \mathcal{H}, \leq, r)$ is a Ramsey space. However, when \mathcal{H} is not necessarily closed, we want to study conditions for \mathcal{H} that will enable us to still make the structure $(\mathcal{R}, \mathcal{H}, \leq, r)$ a Ramsey space and will also allow us to study forcing notions related to \mathcal{H} which will satisfy abstract versions of interesting properties of the corresponding forcing notions in the realm of Ellentuck's space. One such condition is given in Definition 3.7 below.

From now on suppose that for a fixed (\mathcal{R}, \leq, r) , **A1** – **A4** hold and \mathcal{R} is metrically closed. Hence (\mathcal{R}, \leq, r) is a topological Ramsey space. Let $\mathcal{H} \subseteq \mathcal{R}$ be a coideal. The natural definitions of \mathcal{H} -Ramsey and \mathcal{H} -Baire sets are:

Definition 3.2. $\mathcal{X} \subseteq \mathcal{R}$ is **\mathcal{H} -Ramsey** if for every $[a, A] \neq \emptyset$, with $A \in \mathcal{H}$, there exists $B \in [a, A] \cap \mathcal{H}$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \subseteq \mathcal{X}^c$. If for every $[a, A] \neq \emptyset$, there exists $B \in [a, A] \cap \mathcal{H}$ such that $[a, B] \subseteq \mathcal{X}^c$; we say that \mathcal{X} is **\mathcal{H} -Ramsey null**.

Definition 3.3. $\mathcal{X} \subseteq \mathcal{R}$ is **\mathcal{H} -Baire** if for every $[a, A] \neq \emptyset$, with $A \in \mathcal{H}$, there exists $\emptyset \neq [b, B] \subseteq [a, A]$, with $B \in \mathcal{H}$, such that $[b, B] \subseteq \mathcal{X}$ or $[b, B] \subseteq \mathcal{X}^c$. If for every $[a, A] \neq \emptyset$, with $A \in \mathcal{H}$, there exists $\emptyset \neq [b, B] \subseteq [a, A]$, with $B \in \mathcal{H}$, such that $[b, B] \subseteq \mathcal{X}^c$; we say that \mathcal{X} is **\mathcal{H} -meager**.

It is clear that if $\mathcal{X} \subseteq \mathcal{R}$ is \mathcal{H} -Ramsey then \mathcal{X} is \mathcal{H} -Baire.

Definition 3.4. Let $\mathcal{H} \subseteq \mathcal{R}$ be a coideal. Given sets $\mathcal{D}, \mathcal{S} \subseteq \mathcal{H}$, we say that \mathcal{D} is **dense open in \mathcal{S}** if the following hold:

- (1) $(\forall A \in \mathcal{S}) (\exists B \in \mathcal{D}) B \leq A$.
- (2) $(\forall A \in \mathcal{S}) (\forall B \in \mathcal{D}) [A \leq B \rightarrow A \in \mathcal{D}]$.

Definition 3.5. Given $A \in \mathcal{R}$ and a family $\mathcal{A} = \{A_a\}_{a \in \mathcal{AR} \restriction A} \subseteq \mathcal{R}$, we say that $B \in \mathcal{R}$ is a **diagonalization** of \mathcal{A} if for every $a \in \mathcal{AR} \restriction B$ we have $[a, B] \subseteq [a, A_a]$.

Definition 3.6. Given $A \in \mathcal{H}$ and a collection $\mathcal{D} = \{\mathcal{D}_a\}_{a \in \mathcal{AR} \restriction A}$ such that each \mathcal{D}_a is dense open in $\mathcal{H} \cap [\text{depth}_A(a), A]$, we say that $B \in \mathcal{R}$ is a **diagonalization** of \mathcal{D} if there exists a family $\mathcal{A} = \{A_a\}_{a \in \mathcal{AR} \restriction A}$, with $A_a \in \mathcal{D}_a$, such that B is a diagonalization of \mathcal{A} .

Definition 3.7. We say that a coideal $\mathcal{H} \subseteq \mathcal{R}$ is **semiselective** if for every $A \in \mathcal{H}$, every collection $\mathcal{D} = \{\mathcal{D}_a\}_{a \in \mathcal{AR} \upharpoonright A}$ such that each \mathcal{D}_a is dense open in $\mathcal{H} \cap [\text{depth}_A(a), A]$ and every $B \in \mathcal{H} \upharpoonright A$, there exists $C \in \mathcal{H}$ such that C is a diagonalization of \mathcal{D} and $C \leq B$.

Our goal in this section is to prove that the families of \mathcal{H} -Ramsey sets and \mathcal{H} -Baire sets coincide, if \mathcal{H} is semiselective (see Theorem 3.12 below). The following **combinatorial forcing** will be used: Fix $\mathcal{F} \subseteq \mathcal{AR}$. We say that $A \in \mathcal{H}$ **accepts** $a \in \mathcal{AR}$ if for every $B \in \mathcal{H} \cap [\text{depth}_A(a), A]$ there exists $n \in \mathbb{N}$ such that $r_n(B) \in \mathcal{F}$. We say that A **rejects** a if for all $B \in [\text{depth}_A(a), A] \cap \mathcal{H}$, B does not accept a ; and we say that A **decides** a if A either accepts or rejects a .

Lemma 3.8. *The combinatorial forcing has the following properties:*

- (1) *If A both accepts and rejects a then $[a, A] = \emptyset$.*
- (2) *If A accepts a then every $B \in \mathcal{H} \upharpoonright A$ with $[a, B] \neq \emptyset$ accepts a .*
- (3) *If A rejects a , then every $B \in \mathcal{H} \upharpoonright A$ with $[a, B] \neq \emptyset$ rejects a .*
- (4) *For every $A \in \mathcal{H}$ and every $a \in \mathcal{AR} \upharpoonright A$ there exists $B \in [a, A] \cap \mathcal{H}$ which decides a .*
- (5) *If A accepts a then A accepts every $b \in r_{|a|+1}([a, A])$.*
- (6) *If A rejects a then there exists $B \in [a, A] \cap \mathcal{H}$ such that A does not accept any $b \in r_{|a|+1}([a, B])$.*

Proof. (1)–(5) follows from the definitions. Let us prove (6):

Suppose that A rejects a . Let $\mathcal{O} = \{b \in \mathcal{AR} : A \text{ accepts } b\}$. By **A.4** mod \mathcal{H} , there exists $B \in \mathcal{H} \cap [a, A]$ such that $r_{|a|+1}[a, B] \subseteq \mathcal{O}$ or $r_{|a|+1}[a, B] \subseteq \mathcal{O}^c$. If the first alternative holds then take $C \in \mathcal{H} \cap [a, B]$. Let $b = r_{|a|+1}(C)$. Then $b \in \mathcal{O}$ and therefore A accepts b . Since $C \in [b, A]$ then there exists n such that $r_n(C) \in \mathcal{F}$. Therefore B accepts a , because C is arbitrary. But this contradicts that A rejects a . Hence, $r_{|a|+1}[a, B] \subseteq \mathcal{O}^c$ and we are done. \square

Claim. *Given $A \in \mathcal{H}$, there exists $D \in \mathcal{H} \upharpoonright A$ which decides every $b \in \mathcal{AR} \upharpoonright D$.*

Proof. For every $a \in \mathcal{AR} \upharpoonright A$ define

$$\mathcal{D}_a = \{C \in \mathcal{H} \cap [\text{depth}_A(a), A] : C \text{ decides } a\}$$

By parts 2, 3 and 4 of Lemma 3.8 each \mathcal{D}_a is dense open in $\mathcal{H} \cap [\text{depth}_A(a), A]$. By semiselectivity, there exists $D \in \mathcal{H} \upharpoonright A$ which diagonalizes the collection $(\mathcal{D}_a)_{a \in \mathcal{AR} \upharpoonright A}$. By parts 2 and 3 of Lemma 3.8, D decides every $a \in \mathcal{AR} \upharpoonright D$. \square

The following is an abstract version of the semiselective Galvin lemma (see [11, 8]).

Lemma 3.9 (Semiselective abstract Galvin's lemma). *Given $\mathcal{F} \subseteq \mathcal{AR}$, a semiselective coideal $\mathcal{H} \subseteq \mathcal{R}$, and $A \in \mathcal{H}$, there exists $B \in \mathcal{H} \upharpoonright A$ such that one of the following holds:*

- (1) $\mathcal{AR} \upharpoonright B \cap \mathcal{F} = \emptyset$, or
- (2) $\forall C \in [\emptyset, B] (\exists n \in \mathbb{N}) (r_n(C) \in \mathcal{F})$.

Proof. Consider D as in the Claim. If D accepts \emptyset part (2) holds and we are done. So assume that D rejects \emptyset and for $a \in \mathcal{AR} \upharpoonright D$ define

$$\mathcal{D}_a = \{C \in \mathcal{H} \cap [\text{depth}_A(a), D] : C \text{ rejects every } b \in r_{|a|+1}([a, C])\}$$

if D rejects a , and $\mathcal{D}_a = \mathcal{H} \cap [\text{depth}_A(a), D]$, otherwise. By parts 3 and 6 of Lemma 3.8 each \mathcal{D}_a is dense open in $\mathcal{H} \cap [\text{depth}_A(a), D]$. By semiselectivity, choose $B \in \mathcal{H} \upharpoonright D$ such that

for all $a \in \mathcal{AR} \restriction B$ there exists $C_a \in \mathcal{D}_a$ with $[a, B] \subseteq [a, C_a]$. For every $a \in \mathcal{AR} \restriction B$, C_a rejects all $b \in r_{|a|+1}([a, C_a])$. So B rejects all $b \in r_{|a|+1}([a, B])$: given one such b , choose any $\hat{B} \in \mathcal{H} \cap [b, B]$. Then $\hat{B} \in \mathcal{H} \cap [b, C_a]$. Therefore, since C_a rejects b , \hat{B} does not accept b .

Hence, B satisfies that $\mathcal{AR} \restriction B \cap \mathcal{F} = \emptyset$. This completes the proof of the Lemma. \square

Notation. $\mathcal{AR} \restriction [a, B] = \{b \in \mathcal{AR} : a \sqsubseteq b \text{ \& } (\exists n \geq |a|)(\exists C \in [a, B]) b = r_n(C)\}$.

In a similar way we can prove the following generalization of lemma 3.9:

Lemma 3.10. *Given a semiselective coideal \mathcal{H} of \mathcal{R} , $\mathcal{F} \subseteq \mathcal{AR}$, $A \in \mathcal{H}$ and $a \in \mathcal{AR} \restriction A$, there exists $B \in \mathcal{H} \cap [a, A]$ such that one of the following holds:*

- (1) $\mathcal{AR} \restriction [a, B] \cap \mathcal{F} = \emptyset$, or
- (2) $\forall C \in [a, B] (\exists n \in \mathbb{N}) (r_n(C) \in \mathcal{F})$.

Now, we give an application of Lemma 3.9. Recall from Equation 2 that the basic metric open subsets of \mathcal{R} are of the form $[b] = \{A \in \mathcal{R} : b \sqsubset A\}$, where $b \sqsubset A$ means $(\exists n \in \mathbb{N}) (r_n(A) = b)$.

Theorem 3.11. *Suppose that $\mathcal{H} \subseteq \mathcal{R}$ is a semiselective coideal. Then the metric open subsets of \mathcal{R} are \mathcal{H} -Ramsey.*

Proof. Let \mathcal{X} be a metric open subset of \mathcal{R} and fix a nonempty $[a, A]$ with $A \in \mathcal{H}$. Without a loss of generality, we can assume $a = \emptyset$. Since \mathcal{X} is open, there exists $\mathcal{F} \subseteq \mathcal{AR}$ such that $\mathcal{X} = \bigcup_{b \in \mathcal{F}} [b]$. Let $B \in \mathcal{H} \restriction A$ be as in Lemma 3.9. If (1) holds then $[0, B] \subseteq \mathcal{X}^c$ and if (2) holds then $[0, B] \subseteq \mathcal{X}$. \square

The following is one of the main results of this work.

Theorem 3.12. *If $\mathcal{H} \subseteq \mathcal{R}$ is a semiselective coideal then $\mathcal{X} \subseteq \mathcal{R}$ is \mathcal{H} -Ramsey iff \mathcal{X} is \mathcal{H} -Baire*

Proof. Let \mathcal{X} be a \mathcal{H} -Baire subset of \mathcal{R} and consider $A \in \mathcal{H}$. As before, we only proof the result for $[\emptyset, A]$ without a loss of generality. For $a \in \mathcal{AR} \restriction A$ define

$$\begin{aligned} \mathcal{D}_a = \{B \in [\text{depth}_A(a), A] \cap \mathcal{H} : [a, B] \subseteq \mathcal{X} \text{ or } [a, B] \subseteq \mathcal{X}^c \\ \text{or } [(\forall C \in [a, B]) [a, C] \cap \mathcal{X} \neq \emptyset \text{ and } [a, C] \cap \mathcal{X}^c \neq \emptyset]\} \end{aligned}$$

It is easy to see that each \mathcal{D}_a is dense open in $\mathcal{H} \cap [\text{depth}_A(a), A]$. By semiselectivity, choose $B \in \mathcal{H} \restriction A$ which diagonalizes the collection $(\mathcal{D}_a)_{a \in \mathcal{AR} \restriction A}$. Let $\mathcal{F}_0 = \{a \in \mathcal{AR} \restriction A : [a, B] \subseteq \mathcal{X}\}$ and $\mathcal{F}_1 = \{a \in \mathcal{AR} \restriction A : [a, B] \subseteq \mathcal{X}^c\}$. Consider $B_0 \in \mathcal{H} \restriction B$ as in Lemma 3.9 applied to \mathcal{F}_0 and B . If (2) of Lemma 3.9 holds then $[\emptyset, B_0] \subseteq \mathcal{X}$ and we are done. So assume that (1) holds. That is, $\mathcal{AR} \restriction B_0 \cap \mathcal{F}_0 = \emptyset$. Now consider B_1 as in Lemma 3.9 applied to \mathcal{F}_1 and B_0 . Again, if (2) holds then $[\emptyset, B_1] \subseteq \mathcal{X}^c$ and we are done. Notice that $\mathcal{AR}(B_1) \cap \mathcal{F}_1 \neq \emptyset$ because $\mathcal{AR} \restriction B_1 \cap \mathcal{F}_0 = \emptyset$ and \mathcal{X} is \mathcal{H} -Baire. So (2) holds. This concludes the proof. \square

Remark 3.1. In virtue of Theorem 3.12, if \mathcal{H} is a semiselective coideal then $(\mathcal{R}, \mathcal{H}, \leq, r)$ is a Ramsey space. It should be clear that the axioms **A1** - **A4** for general Ramsey spaces are satisfied (see Section 4.2 in [24]) from the definition of coideal and from the fact that (\mathcal{R}, \leq, r) satisfies axioms **A1** - **A4** for topological Ramsey spaces. But we are using semiselectivity (and the fact that \mathcal{R} is closed) instead of asking that \mathcal{H} be closed. We will get more insight into the forcing notion (\mathcal{H}, \leq^*) and other related forcing notions in this way.

The following is a local version of Theorem 1.6 from [17], which is an abstract version of Ramsey's Theorem [22]:

Theorem 3.13. *Suppose that $\mathcal{H} \subseteq \mathcal{R}$ is a semiselective coideal. Then, given a partition $f: \mathcal{AR}_2 \rightarrow \{0, 1\}$ and $A \in \mathcal{H}$, there exists $B \in \mathcal{H} \upharpoonright A$ such that f is constant on $\mathcal{AR}_2(B)$.*

Proof. Let f be the partition $\mathcal{AR}_2 = \mathcal{C}_0 \cup \mathcal{C}_1$, and consider $A \in \mathcal{H}$. Define

$$\mathcal{D}_a = \{B \in [\text{depth}_A(a), A] \cap \mathcal{H} : f \text{ is constant on } r_2[a, B]\}$$

if $a \in \mathcal{AR}_1 \upharpoonright A$ and $\mathcal{D}_a = \mathcal{H} \cap [\text{depth}_A(a), A]$, otherwise. Using **A.4** mod \mathcal{H} in the case $a \in \mathcal{AR}_1 \upharpoonright A$, it is easy to prove that each \mathcal{D}_a is dense open in $\mathcal{H} \cap [\text{depth}_A(a), A]$. By semiselectivity, there exists $B_1 \in \mathcal{H} \upharpoonright A$ which diagonalizes the collection $(\mathcal{D}_a)_{a \in \mathcal{AR}_1 \upharpoonright A}$. Notice that for every $a \in \mathcal{AR}_1 \upharpoonright B_1$, there exists $i_a \in \{0, 1\}$ such that $r_2[a, B_1] \subseteq \mathcal{C}_{i_a}$. Now, consider the partition $g: \mathcal{AR}_1 \rightarrow \{0, 1\}$ defined by $g(a) = i_a$ if $a \in \mathcal{AR}_1 \upharpoonright B_1$. By **A.4** mod \mathcal{H} there exists $B \in \mathcal{H} \cap [0, B_1]$ such that g is constant on $r_1[0, B] = \mathcal{AR}_1(B)$. But $B \leq B_1 \leq A$, so B is as required. \square

Definition 3.14. A coideal $\mathcal{H} \subseteq \mathcal{R}$ is Ramsey if for every integer $n \geq 2$, every partition $f: \mathcal{AR}_n \rightarrow \{0, 1\}$ and $A \in \mathcal{H}$, there exists $B \in \mathcal{H} \upharpoonright A$ such that f is constant on $\mathcal{AR}_n(B)$.

Proceeding in a similar way, using induction, we can prove the following generalization of Theorem 3.13:

Theorem 3.15. *Every semiselective coideal $\mathcal{H} \subseteq \mathcal{R}$ is Ramsey.*

4. THE SOUSLIN OPERATION

Definition 4.1. The result of applying the **Souslin operation** to a family $(\mathcal{X}_a)_{a \in \mathcal{AR}}$ of subsets of \mathcal{R} is:

$$\bigcup_{A \in \mathcal{R}} \bigcap_{n \in \mathbb{N}} \mathcal{X}_{r_n(A)}$$

The goal of this section is to show that the family of \mathcal{H} -Ramsey subsets of \mathcal{R} is closed under the Souslin operation when \mathcal{H} is a semiselective coideal.

Lemma 4.2. *If $\mathcal{H} \subseteq \mathcal{R}$ is a semiselective coideal of \mathcal{R} then the families of \mathcal{H} -Ramsey and \mathcal{H} -Ramsey null subsets of \mathcal{R} are closed under countable unions.*

Proof. Fix $[a, A]$ with $A \in \mathcal{H}$. We will suppose that $a = \emptyset$ without a loss of generality. Let $(\mathcal{X}_n)_{n \in \mathbb{N}}$ be a sequence of \mathcal{H} -Ramsey null subsets of \mathcal{R} . Define for $a \in \mathcal{AR} \upharpoonright A$

$$\mathcal{D}_a = \{B \in \mathcal{H} \cap [a, A] : [a, B] \subseteq \mathcal{X}_n^c \ \forall n \leq |a|\}$$

Every \mathcal{D}_a is dense open in $\mathcal{H} \cap [a, A]$, so let $B \in \mathcal{H} \upharpoonright A$ be a diagonalization of $(\mathcal{D}_a)_a$. Then $[\emptyset, B] \subseteq \bigcap_n \mathcal{X}_n^c$. Thus, $\bigcup_n \mathcal{X}_n$ is \mathcal{H} -Ramsey null. Now, suppose that $(\mathcal{X}_n)_{n \in \mathbb{N}}$ is a sequence of \mathcal{H} -Ramsey subsets of \mathcal{R} . If there exists $B \in \mathcal{H} \upharpoonright A$ such that $[\emptyset, B] \subseteq \mathcal{X}_n$ for some n , we are done. Otherwise, using an argument similar to the one above, we prove that $\bigcup_n \mathcal{X}_n$ is \mathcal{H} -Ramsey null. \square

Let

$$(10) \quad \text{Exp}(\mathcal{H}) = \{[n, A] : n \in \mathbb{N}, A \in \mathcal{H}\}.$$

Definition 4.3. We say $\mathcal{X} \subseteq \mathcal{R}$ is $Exp(\mathcal{H})$ –**nowhere dense** if every member of $Exp(\mathcal{H})$ has a subset in $Exp(\mathcal{H})$ that is disjoint from \mathcal{X} .

Notice that every \mathcal{H} –Ramsey null set is $Exp(\mathcal{H})$ –nowhere dense. And every $Exp(\mathcal{H})$ –nowhere dense is \mathcal{H} –meager. Thus, if \mathcal{H} is semiselective, every $Exp(\mathcal{H})$ –nowhere dense is \mathcal{H} –Ramsey. But if \mathcal{X} is both $Exp(\mathcal{H})$ –nowhere dense and \mathcal{H} –Ramsey, it has to be \mathcal{H} –Ramsey null. As a consequence of Lemma 4.2 and Theorem 3.12 we have:

Corollary 4.1. If $\mathcal{H} \subseteq \mathcal{R}$ is a semiselective coideal of \mathcal{R} and $\mathcal{X} \subseteq \mathcal{R}$ then the following are equivalent:

- (1) \mathcal{X} is \mathcal{H} –Ramsey null
- (2) \mathcal{X} is $Exp(\mathcal{H})$ –nowhere dense.
- (3) \mathcal{X} is $Exp(\mathcal{H})$ –meager (i. e. countable union of $Exp(\mathcal{H})$ –nowhere dense sets).
- (4) \mathcal{X} is \mathcal{H} –meager.

□

Given a set X , say that two subsets A, B of X are “compatible” with respect to a family \mathcal{F} of subsets of X if there exists $C \in \mathcal{F}$ such that $C \subseteq A \cap B$. And \mathcal{F} is M –like if for $\mathcal{G} \subseteq \mathcal{F}$ with $|\mathcal{G}| < |\mathcal{F}|$, every member of \mathcal{F} which is not compatible with any member of \mathcal{G} is compatible with $X \setminus \bigcup \mathcal{G}$. A σ –algebra \mathcal{A} of subsets of X together with a σ –ideal $\mathcal{A}_0 \subseteq \mathcal{A}$ is a *Marczewski pair* if for every $A \subseteq X$ there exists $\Phi(A) \in \mathcal{A}$ such that $A \subseteq \Phi(A)$ and for every $B \subseteq \Phi(A) \setminus A$, $B \in \mathcal{A} \Rightarrow B \in \mathcal{A}_0$. The following is a well known fact:

Theorem 4.4 (Marczewski). *Every σ –algebra of sets which together with a σ –ideal is a Marczewski pair, is closed under the Souslin operation.*

□

Let \mathcal{H} be a semiselective coideal of \mathcal{R} . Then we have:

Proposition 4.5. *The family $Exp(\mathcal{H})$ is M –like.*

Proof. Consider $\mathcal{B} \subseteq Exp(\mathcal{H})$ with $|\mathcal{B}| < |Exp(\mathcal{H})| = 2^{\aleph_0}$ and suppose that $[a, A]$ is not compatible with any member of \mathcal{B} , i. e. for every $[b, B] \in \mathcal{B}$, $[b, B] \cap [a, A]$ does not contain any member of $Exp(\mathcal{H})$. We claim that $[a, A]$ is compatible with $\mathcal{R} \setminus \bigcup \mathcal{B}$. In fact:

Since $|\mathcal{B}| < 2^{\aleph_0}$, by Lemmas 3.10 and sigmaideal, $\bigcup \mathcal{B}$ is \mathcal{H} –Ramsey and therefore \mathcal{H} –Baire, by Theorem 3.12. So, there exist $[b, B] \subseteq [a, A]$ with $B \in \mathcal{H}$ such that:

- (1) $[b, B] \subseteq \bigcup \mathcal{B}$ or
- (2) $[b, B] \subseteq \mathcal{R} \setminus \bigcup \mathcal{B}$

(1) is not possible because $[a, A]$ is not compatible with any member of \mathcal{B} . And (2) says that $[a, A]$ is compatible with $\mathcal{R} \setminus \bigcup \mathcal{B}$. This completes the proof. □

Proposition 4.5 says that the family of \mathcal{H} –Ramsey subsets of \mathcal{R} together with the family of \mathcal{H} –Ramsey null subsets of \mathcal{R} is a Marczewski pair. Thus, by theorem 4.4, we obtain the following:

Theorem 4.6. *The family of \mathcal{H} –Ramsey subsets of \mathcal{R} is closed under the Souslin operation.*

□

5. ABSTRACT SELECTIVITY

Definition 5.1. Given $A \in \mathcal{H}$ and $\mathcal{A} = (A_a)_{a \in \mathcal{AR} \upharpoonright A} \subseteq \mathcal{H} \upharpoonright A$ with $[a, A_a] \neq \emptyset$ for all a , we say that \mathcal{A} is **filtered by \leq** if for every $a, b \in \mathcal{AR} \upharpoonright A$ there exists $c \in \mathcal{AR} \upharpoonright A$ such that $A_c \leq A_a$ and $A_c \leq A_b$.

Definition 5.2. A coideal $\mathcal{H} \subseteq \mathcal{R}$ is **selective** if given $A \in \mathcal{H}$, for every $\mathcal{A} = (A_a)_{a \in \mathcal{AR} \upharpoonright A} \subseteq \mathcal{H} \upharpoonright A$ filtered by \leq such that $[a, A_a] \neq \emptyset$ for all a , there exists $B \in \mathcal{H} \upharpoonright A$ which diagonalizes \mathcal{A} .

Lemma 5.3. Given a coideal \mathcal{H} of \mathcal{R} and $A \in \mathcal{H}$, for every $(\mathcal{D}_a)_{a \in \mathcal{AR} \upharpoonright A}$ such that each \mathcal{D}_a is dense open in $\mathcal{H} \cap [a, A]$ there exists $(A_a)_{a \in \mathcal{AR} \upharpoonright A}$ filtered by \leq such that $A_a \in \mathcal{D}_a$ for all $a \in \mathcal{AR} \upharpoonright A$.

Proof. For every $k \in \mathbb{N}$, list

$$\mathcal{A}_k = \{a_1^1, a_2^1, \dots, a_{n_k}^1\} = \{a \in \mathcal{AR} \upharpoonright A : \text{depth}_A(a) = k\}$$

(every \mathcal{A}_k is finite by **A2**). Since each \mathcal{D}_a is dense open in $\mathcal{H} \cap [a, A]$, using **A3 mod \mathcal{H}** we can choose $A^{1,1} \in \mathcal{D}_{a_1^1} \upharpoonright A$, $A^{1,2} \in \mathcal{D}_{a_2^1} \upharpoonright A^{1,1}$, \dots , $A^{1,n_1} \in \mathcal{D}_{a_{n_1}^1} \upharpoonright A^{1,n_1-1}$. Again, we can choose $A^{2,1} \in \mathcal{D}_{a_1^2} \upharpoonright A^{1,n_1}$, $A^{2,2} \in \mathcal{D}_{a_2^2} \upharpoonright A^{2,1}$, \dots , $A^{2,n_2} \in \mathcal{D}_{a_{n_2}^2} \upharpoonright A^{2,n_2-1}$. And so on. Then $(A^{i,j})_{ij}$ is as required. \square

Proposition 5.4. If $\mathcal{H} \subseteq \mathcal{R}$ is a selective coideal then \mathcal{H} is semiselective.

Proof. Consider $A \in \mathcal{H}$ and let $\mathcal{D} = (\mathcal{D}_a)_{a \in \mathcal{AR} \upharpoonright A}$ be such that each \mathcal{D}_a is dense open in $\mathcal{H} \cap [a, A]$. It is clear that if $\hat{B} \in \mathcal{H} \upharpoonright A$ then \mathcal{D}_a is dense open in $\mathcal{H} \cap [a, \hat{B}]$, for all $a \in \hat{B}$. Using lemma 5.3 we can build $\mathcal{A} = (A_a)_{a \in \mathcal{AR} \upharpoonright \hat{B}}$ filtered by \leq such that $A_a \in \mathcal{D}_a$ for every $a \in \mathcal{AR} \upharpoonright \hat{B}$. By selectivity, there exists $B \in \mathcal{H} \upharpoonright \hat{B}$ which diagonalizes \mathcal{A} . \square

6. ABSTRACT SEMISELECTIVITY AND FORCING

6.1. Forcing with (\mathcal{H}, \leq^*) .

Notation. We will borrow the the following notation from Section 2 of [17]. For $A, B \in \mathcal{R}$, write $A \leq^* B$ if there exists $a \in \mathcal{AR} \upharpoonright A$ such that $[a, A] \subseteq [a, B]$. In this case we say that A is an *almost-reduction* of B . This is a generalization of *almost-inclusion* and *almost-condensation* (see [1]). In [17], it is proved that (\mathcal{R}, \leq^*) is reflexive and transitive.

In this section we will describe the main properties of the forcing notion (\mathcal{H}, \leq^*) , for a semiselective \mathcal{H} . To do so, we will need to consider a special type of coideal $\mathcal{U} \subseteq \mathcal{R}$:

Definition 6.1. Given $\mathcal{U} \subseteq \mathcal{R}$, we say that \mathcal{U} is an **ultrafilter** if it satisfies the following:

- (a) \mathcal{U} is a *filter* on (\mathcal{R}, \leq) . That is:
 - (1) For all $A, B \in \mathcal{R}$, if $A \in \mathcal{U}$ and $A \leq B$ then $B \in \mathcal{U}$.
 - (2) For all $A, B \in \mathcal{U}$ and $a \in \mathcal{AR}$, if $[a, A] \neq \emptyset$ and $[a, B] \neq \emptyset$, then there exists $C \in \mathcal{U}$ such that $C \in [a, A] \cap [a, B]$. In particular, for all $A, B \in \mathcal{U}$, there exists $C \in \mathcal{U}$ such that $C \leq A$ and $C \leq B$.
- (b) If $\mathcal{U}' \subseteq \mathcal{R}$ is a filter on (\mathcal{R}, \leq) and $\mathcal{U} \subseteq \mathcal{U}'$ then $\mathcal{U}' = \mathcal{U}$. That is, \mathcal{U} is a *maximal filter* on (\mathcal{R}, \leq) .
- (c) (**A3 mod \mathcal{U}**) For all $A \in \mathcal{U}$ and $a \in \mathcal{AR} \upharpoonright A$, the following holds:
 - $[a, B] \neq \emptyset$ for all $B \in [\text{depth}_A(a), A] \cap \mathcal{U}$.

- If $B \in \mathcal{U} \restriction A$ and $[a, B] \neq \emptyset$ then there exists $A' \in [\text{depth}_A(a), A] \cap \mathcal{U}$ such that $\emptyset \neq [a, A'] \subseteq [a, B]$.

Remark 6.1. In [25], a similar abstract definition of ultrafilter is used. But the ultrafilters used in [25] do not satisfy part (c) of Definition 6.1.

It turns out that every such ultrafilter $\mathcal{U} \subseteq \mathcal{R}$ also satisfies the following very useful condition and therefore it is a coideal.

- (d) (**A4** mod \mathcal{U}) Let $A \in \mathcal{U}$ and $a \in \mathcal{AR} \restriction A$ be given. For all $\mathcal{O} \subseteq \mathcal{AR}_{|a|+1}$ then there exists $B \in [\text{depth}_A(a), A] \cap \mathcal{U}$ such that $r_{|a|+1}[a, B] \subseteq \mathcal{O}$ or $r_{|a|+1}[a, B] \cap \mathcal{O} = \emptyset$.

Lemma 6.2. *If $\mathcal{H} \subseteq \mathcal{R}$ is a semiselective coideal then (\mathcal{H}, \leq^*) is σ -distributive.*

Proof. For every $n \in \mathbb{N}$, let $\mathcal{D}_n \subseteq \mathcal{H}$ be dense open in (\mathcal{H}, \leq^*) . Fix $A \in \mathcal{H}$. For all $a \in \mathcal{AR} \restriction A$, the set $\mathcal{D}_a = \{B \in \mathcal{H} \cap [\text{depth}_A(a), A] : B \in \mathcal{D}_{|a|}\}$ is dense open in $\mathcal{H} \cap [\text{depth}_A(a), A]$: Fix $a \in \mathcal{AR} \restriction A$. Obviously, if $B \in \mathcal{D}_a$ and $B' \in \mathcal{H} \cap [\text{depth}_A(a), A]$ is such that $B' \leq B$ then $B' \in \mathcal{D}_a$. On the other hand, given $C \in \mathcal{H} \cap [\text{depth}_A(a), A]$, choose $B_a \in \mathcal{D}_{|a|}$ such that $B_a \leq^* C$. Then there exists $b \in \mathcal{AR} \restriction B_a$ such that $[b, B_a] \subseteq [b, C]$. We will assume that $\text{depth}_C(b) \geq \text{depth}_C(a) = \text{depth}_A(a)$ (otherwise, let $m = |b| + \text{depth}_C(a)$ and choose $D \in [b, B_a]$. Let $\hat{b} = r_{m+1}(D)$. Then $[\hat{b}, B_a] \subseteq [\hat{b}, C]$ and $\text{depth}_C(\hat{b}) \geq \text{depth}_C(a)$). By **A3** mod \mathcal{H} , choose $B \in \mathcal{H} \cap [\text{depth}_C(b), C]$ such that $\emptyset \neq [b, B] \subseteq [b, B_a]$. So $B \leq^* B_a$ and therefore $B \in \mathcal{D}_{|a|}$. Notice also that since $\text{depth}_C(b) \geq \text{depth}_C(a) = \text{depth}_A(a)$ then $B \in \mathcal{H} \cap [\text{depth}_A(a), C] \subseteq \mathcal{H} \cap [\text{depth}_A(a), A]$. This implies that $B \leq C$ and $B \in \mathcal{D}_a$. This completes the proof that \mathcal{D}_a is dense open. Let $B \in \mathcal{H} \restriction A$ be a diagonalization of $\{\mathcal{D}_a\}_{a \in \mathcal{AR} \restriction A}$. Then there exists a family $\{A_a\}_{a \in \mathcal{AR} \restriction A}$ with $A_a \in \mathcal{D}_a$ such that $[a, B] \subseteq [a, A_a]$ for all $a \in \mathcal{AR} \restriction B$. This means that $B \leq^* A_a$ for all $a \in \mathcal{AR} \restriction B$. Therefore, $B \in \mathcal{D}_{|a|}$ for all $a \in \mathcal{AR} \restriction B$. That is, $B \in \bigcap_n \mathcal{D}_n$. This completes the proof. \square

Lemma 6.3. *Let \mathcal{R} be a topological Ramsey space and $\mathcal{H} \subseteq \mathcal{R}$ be a semiselective coideal. Forcing with (\mathcal{H}, \leq^*) adds no new elements of $\mathcal{AR}^{\mathbb{N}}$ (in particular, no new elements of \mathcal{R} or \mathcal{H}), and if \mathcal{U} is the (\mathcal{H}, \leq^*) -generic filter over some ground model V , then \mathcal{U} is a Ramsey ultrafilter in $V[\mathcal{U}]$.*

Proof. Since (\mathcal{H}, \leq^*) is σ -distributive, the fact that forcing with (\mathcal{H}, \leq^*) adds no new elements of $\mathcal{AR}^{\mathbb{N}}$ follows by a standard argument. See for instance [13], Theorem 2.10. Let \mathcal{U} be the (\mathcal{H}, \leq^*) -generic filter over some ground model V . By genericity, \mathcal{U} is a maximal filter. Also by genericity, **A3** (for the space \mathcal{R}) and Theorem 3.15, we have that **A3** mod \mathcal{U} holds (and therefore, \mathcal{U} satisfies Definition 6.1) and \mathcal{U} is Ramsey. \square

Lemma 6.4. *Let \mathcal{U} be the (\mathcal{H}, \leq^*) -generic filter over some ground model V . Then \mathcal{U} is selective in $V[\mathcal{U}]$.*

Proof. In $V[\mathcal{U}]$, fix $A \in \mathcal{U}$ and let $\{A_a\}_{a \in \mathcal{AR} \restriction A}$ be a collection of elements of \mathcal{U} such that $[a, A_a] \neq \emptyset$, for all $a \in \mathcal{AR}$. Given an integer $n > 0$, define $f : \mathcal{AR}_{n+1} \rightarrow \{0, 1\}$ as $f(b) = 1$ if and only if $[b, A_{r_n(b)}] \neq \emptyset$. By Lemma 6.3, \mathcal{U} is a Ramsey ultrafilter in $V[\mathcal{U}]$ so there exist $B \in \mathcal{U}$ such that $B \leq A$ and f is constant in $\mathcal{AR}_{n+1} \restriction B$. Take an arbitrary $a \in \mathcal{AR}_n \restriction B$. Notice that there exists $C \in \mathcal{U}$ such that $C \leq B$, $C \leq A_a$, and $[a, C] \neq \emptyset$, by part (a)(2) of Definition 6.1. For any $b \in r_{n+1}[a, C]$ we have $f(b) = 1$, and therefore f takes

constant value 1 in $\mathcal{AR}_{n+1} \restriction B$. Since a is arbitrary, this implies that for all $a \in \mathcal{AR}_n \restriction B$, $r_{n+1}[a, B] \subseteq r_{n+1}[a, A_a]$.

Now, recall that $\mathcal{U} \subseteq \mathcal{H}$. So the above reasoning implies that for every integer $n > 0$, the set

$$\mathcal{E}_n = \{B \in \mathcal{H} : (\forall a \in \mathcal{AR}_n \restriction B) \ r_{n+1}[a, B] \subseteq r_{n+1}[a, A_a]\}$$

is dense open in (\mathcal{H}, \leq^*) . So, by genericity of \mathcal{U} and σ -distributivity of (\mathcal{H}, \leq^*) , we can choose $B \in \mathcal{U} \cap \bigcap_n \mathcal{E}_n$. So for every $a \in \mathcal{AR} \restriction B$ we have $r_{|a|+1}[a, B] \subseteq r_{|a|+1}[a, A_a]$. By **A1**, this implies that $[a, B] \subseteq [a, A_a]$, for every $a \in \mathcal{AR} \restriction B$. Therefore, B is a diagonalization of $(A_a)_{a \in \mathcal{AR}}$. This completes the proof. \square

Lemma 6.5. *Suppose \mathcal{H} is not semiselective. Let \mathcal{U} be (\mathcal{H}, \leq^*) -generic filter over some ground model V . Then \mathcal{U} is not selective in $V[\mathcal{U}]$.*

Proof. Since \mathcal{H} is not semiselective, there exist $A \in \mathcal{H}$ and a collection $(\mathcal{D}_a)_{a \in \mathcal{AR} \restriction A}$ such that \mathcal{D}_a is dense in $\mathcal{H} \cap [\text{depth}_A(a), A]$, for every $a \in \mathcal{AR} \restriction A$, with no diagonalization in \mathcal{H} . In $V[\mathcal{U}]$, it turns out that each \mathcal{D}_a is dense in $(\mathcal{H} \restriction A, \leq^*)$. Proceeding as in Lemma 5.3, find a collection $(A_a)_{a \in \mathcal{AR} \restriction A}$ filtered by \leq^* and such that $A_a \in \mathcal{U} \cap \mathcal{D}_a$, for every $a \in \mathcal{AR} \restriction A$. Then the collection $(A_a)_{a \in \mathcal{AR} \restriction A}$ has no diagonalization in \mathcal{U} and therefore \mathcal{U} is not selective. This completes the proof. \square

The following theorem summarizes all the results of this section.

Theorem 6.6. *Let $\mathcal{H} \subseteq \mathcal{R}$ be a coideal. The following are equivalent.*

- (1) \mathcal{H} is semiselective.
- (2) *Forcing with (\mathcal{H}, \leq^*) adds no new elements of $\mathcal{AR}^{\mathbb{N}}$ (in particular, no new elements of \mathcal{R} or \mathcal{H}), and if \mathcal{U} is the (\mathcal{H}, \leq^*) -generic filter over some ground model V , then \mathcal{U} is a selective ultrafilter in $V[\mathcal{U}]$.*

6.2. Forcing with $\mathbb{M}_{\mathcal{H}}$. Let \mathcal{H} be a semiselective coideal. In this section we will study the forcing notion induced by the following poset:

$$\mathbb{M}_{\mathcal{H}} = \{(a, A) : A \in \mathcal{H}, a \in \mathcal{AR} \restriction A\} \cup \{\emptyset\},$$

where $(a, A) \leq (b, B)$ if and only if $[a, A] \subseteq [b, B]$. We say that $\mathbb{M}_{\mathcal{H}}$ is the **Mathias poset** associated to \mathcal{H} . It is said that \mathcal{H} has the **Prikry property** if for every sentence of the forcing language ϕ and every condition $(a, A) \in \mathbb{M}_{\mathcal{H}}$ there exists $B \in [a, A] \cap \mathcal{H}$ such that (a, B) decides ϕ . We say that $x \in \mathcal{R}$ is $\mathbb{M}_{\mathcal{H}}$ -generic over a model V if for every dense open subset $\mathcal{D} \in M$ of $\mathbb{M}_{\mathcal{H}}$, there exists a condition $(a, A) \in \mathcal{D}$ such that $x \in [a, A]$. It is said that \mathcal{H} has the **Mathias property** if it satisfies that if x is $\mathbb{M}_{\mathcal{H}}$ -generic over a model V , then every $y \leq x$ is $\mathbb{M}_{\mathcal{H}}$ -generic over M .

Theorem 6.7. *If $\mathcal{H} \subseteq \mathcal{R}$ is a semiselective coideal then it has the Prikry property.*

Proof. Suppose \mathcal{H} is semiselective and fix a sentence ϕ of the forcing language, and a condition $(a, A) \in \mathbb{M}_{\mathcal{H}}$. For every $b \in \mathcal{AR} \restriction A$ with $b \sqsupseteq a$, let

$$\mathcal{D}_b = \{B \in \mathcal{H} \cap [\text{depth}_A(b), A] : (b, B) \text{ decides } \phi \text{ or } (\forall C \in \mathcal{H} \cap [b, B]) \ (b, C) \text{ does not decide } \phi\}.$$

and set $\mathcal{D}_b = \mathcal{H} \cap [\text{depth}_A(b), A]$, for all $b \in \mathcal{AR} \restriction A$ with $b \not\sqsupseteq a$. Each \mathcal{D}_b is dense open in $\mathcal{H} \cap [\text{depth}_A(b), A]$. Fix a diagonalization $B \in \mathcal{H} \restriction A$. For every $b \in \mathcal{AR} \restriction A$. Let

$$\mathcal{F}_0 = \{b \in \mathcal{AR} \upharpoonright B : a \sqsubseteq b \text{ \& } (b, B) \text{ forces } \phi\},$$

$$\mathcal{F}_1 = \{b \in \mathcal{AR} \upharpoonright B : a \sqsubseteq b \text{ \& } (b, B) \text{ forces } \neg\phi\}.$$

Let $\hat{C} \in \mathcal{H} \upharpoonright B$ as in Lemma 3.10 applied to B and \mathcal{F}_0 . And let $C \in \mathcal{H} \upharpoonright \hat{C}$ be as in Lemma 3.9 applied to \hat{C} and \mathcal{F}_1 . Let us prove that (a, C) decides ϕ . So let (b_0, C_0) and (b_1, C_1) be two different arbitrary extensions of (a, C) . Suppose that (b_0, C_0) forces ϕ and (b_1, C_1) forces $\neg\phi$. Then $b_0 \in \mathcal{F}_0$ and $b_1 \in \mathcal{F}_1$. But $b_0, b_1 \in \mathcal{AR} \upharpoonright C$, so by the choice of C this means that every element of $\mathcal{H} \cap [a, C]$ has an initial segment both in \mathcal{F}_0 and in \mathcal{F}_1 . So there exist two compatible extensions of (a, C) such that one forces ϕ and the other forces $\neg\phi$. A contradiction. So either both (b_0, C_0) and (b_1, C_1) force ϕ or both (b_0, C_0) and (b_1, C_1) force $\neg\phi$. Therefore (a, C) decides ϕ . \square

Now, we will prove that if $\mathcal{H} \subseteq \mathcal{R}$ is semiselective then it has the Mathias property (see Theorem 6.13 below). Given a selective ultrafilter $\mathcal{U} \subset \mathcal{R}$, let $\mathbb{M}_{\mathcal{U}}$ be set of all pairs (a, A) such that $A \in \mathcal{U}$ and $[a, A] \neq \emptyset$. Order $\mathbb{M}_{\mathcal{U}}$ with the same ordering used for $\mathbb{M}_{\mathcal{H}}$. The results contained in the rest of this Section were essentially proved in [17], but they were written (in [17]) for the case $\mathcal{H} = \mathcal{R}$ and using weaker definitions of ultrafilter and selectivity. We adapted the proofs and include them here for completeness.

Definition 6.8. Let $\mathcal{U} \subseteq \mathcal{R}$ be a selective ultrafilter, \mathcal{D} a dense open subset of $\mathbb{M}_{\mathcal{U}}$, and $a \in \mathcal{AR}$. We say that A captures (a, \mathcal{D}) if $A \in \mathcal{U}$, $[a, A] \neq \emptyset$, and for all $B \in [a, A]$ there exists $m > |a|$ such that $(r_m(B), A) \in \mathcal{D}$.

Lemma 6.9. Let $\mathcal{U} \subseteq \mathcal{R}$ be a selective ultrafilter and \mathcal{D} a dense open subset of $\mathbb{M}_{\mathcal{U}}$. Then, for every $a \in \mathcal{AR}$ there exists A which captures (a, \mathcal{D}) .

Proof. Given $a \in \mathcal{AR}$, choose $B \in \mathcal{U}$ such that $[a, B] \neq \emptyset$. We can define a collection $(C_b)_{b \in \mathcal{AR} \upharpoonright B}$, filtered by \leq and with $[b, C_b] \neq \emptyset$, such that for all $b \in \mathcal{AR} \upharpoonright B$ with $a \sqsubseteq b$ either $(b, C_b) \in \mathcal{D}$ or $C_b = B$. By selectivity, let $C \in \mathcal{U} \cap [a, B]$ be a diagonalization of $(C_b)_{b \in \mathcal{AR} \upharpoonright B}$. Then, for all $b \in \mathcal{AR} \upharpoonright B$ with $a \sqsubseteq b$, if there exists a $\hat{C} \in \mathcal{U}$ such that $(b, \hat{C}) \in \mathcal{D}$, we must have $(b, C) \in \mathcal{D}$. Let $\mathcal{X} = \{D \in \mathcal{R} : D \leq C \rightarrow (\exists b \in \mathcal{AR} \upharpoonright D) a \sqsubset b \text{ \& } (b, C) \in \mathcal{D}\}$. \mathcal{X} is a metric open subset of \mathcal{R} and therefore, by Theorem 3.11, it is \mathcal{U} -Ramsey. Take $\hat{C} \in \mathcal{U} \cap [\text{depth}_C(a), C]$ such that $[a, \hat{C}] \subseteq \mathcal{X}$ or $[a, \hat{C}] \cap \mathcal{X} = \emptyset$. We will show that the first alternative holds: Pick $A \in \mathcal{U} \cap [a, \hat{C}]$ and $(a', A') \in \mathcal{D}$ such that $(a', A') \leq (a, A)$. Notice that $a \sqsubseteq a'$ and therefore, as we pointed out at the end of the previous paragraph, we have $(a', C) \in \mathcal{D}$. By the definition of \mathcal{X} , we also have $A' \in \mathcal{X}$. Now choose $A'' \in \mathcal{U} \cap [a, A']$. Then (a', A'') is also in \mathcal{D} and therefore $A'' \in \mathcal{X} \cap [a, \hat{C}]$. This implies $[a, \hat{C}] \subseteq \mathcal{X}$. Finally, that A captures (a, \mathcal{D}) follows from the definition of \mathcal{X} and the fact that $[a, A] \subseteq [a, \hat{C}] \subseteq [a, \hat{C}]$. This completes the proof. \square

Theorem 6.10. Let $\mathcal{U} \subseteq \mathcal{R}$ be a selective ultrafilter, in a given transitive model of $ZF + DCR$. Forcing over M with $\mathbb{M}_{\mathcal{U}}$ adds a generic $g \in \mathcal{R}$ with the property that $g \leq^* A$ for all $A \in \mathcal{U}$. In fact, $B \in \mathcal{R}$ is $\mathbb{M}_{\mathcal{U}}$ -generic over V if and only if $B \leq^* A$ for all $A \in \mathcal{U}$. Also, $V[\mathcal{U}][g] = V[g]$.

Proof. Suppose that $B \in \mathcal{R}$ is $\mathbb{M}_{\mathcal{U}}$ -generic over V . Fix an arbitrary $A \in \mathcal{U}$. The set $\{(c, C) \in \mathbb{M}_{\mathcal{U}} : C \leq^* A\}$ is dense open and is in V : Fix $(a, A') \in \mathbb{M}_{\mathcal{U}}$. Choose $C_0 \in \mathcal{U}$ such that $C_0 \leq A, A'$. Since \mathcal{U} is an ultrafilter, we can choose $C_1 \in \mathcal{U}$ and $n \in \omega$ such that $[n, C_1] \subseteq [a, A'] \cap [1, C_0]$. Let $c = r_n(C_1)$. By **A3** mod \mathcal{U} , there exists $C_2 \in \mathcal{U} \cap [\text{depth}_{A'}(c), A']$ such that $\emptyset \neq [c, C_2] \subseteq [c, C_1]$. It is clear that $[c, C_2] \subseteq [c, A]$ and therefore $C_2 \leq^* A$. Also, since $\text{depth}_{A'}(c) \geq \text{depth}_{A'}(a)$, we have $[a, C_2] \neq \emptyset$. Thus, $(a, C_2) \leq (a, A')$. That is, \mathcal{D} is dense. It is obviously open. So, by genericity, there exists one $(c, C) \in \mathcal{D}$ such that $B \in [c, C]$. Hence $B \leq^* A$.

Now, suppose that $B \in \mathcal{R}$ is such that $B \leq^* A$ for all $A \in \mathcal{U}$, and let \mathcal{D} be a dense open subset of $\mathbb{M}_{\mathcal{U}}$. We need to find $(a, A) \in \mathcal{D}$ such that $B \in [a, A]$. In V , by using Lemma 6.9 iteratively, we can define a sequence $(A_n)_n$ such that $A_n \in \mathcal{U}$, $A_{n+1} \leq A_n$, and A_n captures $(r_n(B), \mathcal{D})$. Since \mathcal{U} is in V and selective, we can choose $A \in \mathcal{U}$, in V , such that $A \leq^* A_n$ for all n . By our assumption on B , we have $B \leq^* A$. So there exists an $a \in \mathcal{AR}$ such that $[a, B] \subseteq [a, A]$. Let $m = \text{depth}_B(a)$. By **A3** mod \mathcal{U} , we can assume that $a = r_m(B) = r_m(A)$, and also that $A \in [r_m(B), A_m]$. Therefore, $B \in [m, A]$ and A captures $(r_m(B), \mathcal{D})$. Hence, the following is true in V :

$$(11) \quad (\forall C \in [m, A])(\exists n > m)((r_n(C), A) \in \mathcal{D}).$$

Let $\mathcal{F} = \{b : (\exists n > m)(b \in r_n[m, A] \text{ \& } (b, A) \notin \mathcal{D})\}$ and give \mathcal{F} the strict end-extension ordering \sqsubset . Then the relation (\mathcal{F}, \sqsubset) is in V , and by equation 11 (\mathcal{F}, \sqsubset) is well-founded. Therefore, by a well-known argument due to Mostowski, equation 11 holds in the universe. Hence, since $B \in [m, A]$, there exists $n > m$ such that $(r_n(B), A) \in \mathcal{D}$. But $B \in [r_n(B), A]$, so B is $\mathbb{M}_{\mathcal{U}}$ -generic over V .

Finally, if g is $\mathbb{M}_{\mathcal{U}}$ -generic over V , then, in V , $\mathcal{U} = \{A \in V : A \in \mathcal{R} \text{ \& } g \leq^* A\}$ and therefore $V[\mathcal{U}][g] = V[g]$. □

Corollary 6.11. *If B is $\mathbb{M}_{\mathcal{U}}$ -generic over some model V and $A \leq B$ then A is also $\mathbb{M}_{\mathcal{U}}$ -generic over V .*

Lemma 6.12. *Let $\mathcal{H} \subseteq \mathcal{R}$ be a semiselective coideal. Consider the forcing notion $\mathbb{P} = (\mathcal{H}, \leq^*)$ and let $\hat{\mathcal{U}}$ be a \mathbb{P} -name for a \mathbb{P} -generic ultrafilter. Then the iteration $\mathbb{P} * \mathbb{M}_{\hat{\mathcal{U}}}$ is equivalent to the forcing $\mathbb{M}_{\mathcal{H}}$.*

Proof. Recall that $\mathbb{P} * \mathbb{M}_{\hat{\mathcal{U}}} = \{(B, (\dot{a}, \dot{A})) : B \in \mathcal{H} \text{ \& } B \vdash (\dot{a}, \dot{A}) \in \mathbb{M}_{\hat{\mathcal{U}}}\}$, with the ordering $(B, (\dot{a}, \dot{A})) \leq (B_0, (\dot{a}_0, \dot{A}_0)) \Leftrightarrow B \leq^* B_0 \text{ \& } (\dot{a}, \dot{A}) \leq (B_0, (\dot{a}_0, \dot{A}_0))$. The mapping $(a, A) \rightarrow (A, (\hat{a}, \hat{A}))$ is a dense embedding (see [14]) from $\mathbb{M}_{\mathcal{H}}$ to $\mathbb{P} * \mathbb{M}_{\hat{\mathcal{U}}}$ (here \hat{a} and \hat{A} are the canonical \mathbb{P} -names for a and A , respectively): It is easy to show that this mapping preserves the order. So, given $(B, (\dot{a}, \dot{A})) \in \mathbb{P} * \mathbb{M}_{\hat{\mathcal{U}}}$, we need to find $(d, D) \in \mathbb{M}_{\mathcal{H}}$ such that $(D, (\hat{d}, \hat{D})) \leq (B, (\dot{a}, \dot{A}))$. Since \mathbb{P} is σ -distributive, there exists $a \in \mathcal{AR}$, $A \in \mathcal{H}$ and $C \leq^* B$ in \mathcal{H} such that $C \vdash_{\mathbb{P}} (\hat{a} = \dot{a} \text{ \& } \hat{A} = \dot{A})$ (so we can assume $a \in \mathcal{AR} \setminus C$). Notice that $(C, (\hat{a}, \hat{A})) \in \mathbb{P} * \mathbb{M}_{\hat{\mathcal{U}}}$ and $(C, (\hat{a}, \hat{A})), (C, (\hat{a}, \hat{A})) \leq (B, (\dot{a}, \dot{A}))$. So, $C \vdash_{\mathbb{P}} \hat{C} \in \hat{\mathcal{U}}$ and $C \vdash_{\mathbb{P}} \hat{A} \in \hat{\mathcal{U}}$. Then, $C \vdash_{\mathbb{P}} (\exists x \in \hat{\mathcal{U}})(x \in [\hat{a}, \hat{A}] \text{ \& } x \in [\hat{a}, \hat{C}])$. So there exists $D \in \mathcal{H}$ such that $D \in [a, A] \cap [a, C]$. Hence, $(D, (\hat{a}, \hat{D})) \leq (B, (\dot{a}, \dot{A}))$. This completes the proof. □

The next Theorem follows immediately from Corollary 6.11 and Lemma 6.12.

Theorem 6.13. *If $\mathcal{H} \subseteq \mathcal{R}$ is a semiselective coideal then it has Mathias property.*

7. SOME APPLICATIONS

7.1. Selectivity and genericity. Let \mathcal{R} be a topological Ramsey space. In this Section we will show that if the existence of a super compact cardinal is consistent then so is the statement “every semiselective ultrafilter $\mathcal{U} \subseteq \mathcal{R}$ is generic over $L(\mathbb{R})$ ”. First, let us state the following definition (see [8], Definition 4.10; and also [9, 10, 23]).

Definition 7.1. Let M be a countable elementary submodel of some structure of the form H_θ which contains a poset \mathcal{P} and a \mathcal{P} -name \hat{r} for a real. Then we say that M is $(L(\mathbb{R}), \mathcal{P})$ -**correct** if for every (M, \mathcal{P}) -generic filter $G \subseteq \mathcal{P} \cap M$ and every formula $\phi(x, \vec{p})$ with parameter \vec{p} in M , the formula $\phi(\text{val}_G(\hat{r}), \vec{p})$ is true in $L(\mathbb{R})$ if and only if there exists a condition in G which forces this. We say that **truth in $L(\mathbb{R})$ is unchangeable by forcing** if the following condition is satisfied: For every poset \mathcal{P} there exists θ large enough so that there exist stationarily many countable elementary submodels M of H_θ which are $(L(\mathbb{R}), \mathcal{P})$ -correct.

We will also need the following two lemmas. For the proof of Lemma 7.2 see [9, 10, 23]. And for the proof of Lemma 7.3 see [8].

Lemma 7.2. *If there exist a supercompact cardinal, then truth in $L(\mathbb{R})$ is unchangeable by forcing.*

Lemma 7.3. *Assume that truth in $L(\mathbb{R})$ is unchangeable by forcing. If E is a ccc space and $f : E \rightarrow \mathbb{R}$ is continuous then for every set of reals \mathcal{X} from $L(\mathbb{R})$, the set $f^{-1}(\mathcal{X})$ has the Baire property.*

Now we are ready to prove the following.

Theorem 7.4. *If there exists a super compact cardinal, then every selective coideal $\mathcal{U} \subseteq \mathcal{R}$ is (\mathcal{R}, \leq^*) -generic over $L(\mathbb{R})$.*

Proof. This is a generalization of Todorcevic’s proof of the corresponding result for $\mathcal{R} = \mathbb{N}^{[\infty]}$ (see Theorem 4.9 of [8]). Let $\mathcal{U} \subseteq \mathcal{R}$ be a selective ultrafilter. Let E be the topological space \mathcal{R} with the topology generated by the family $\text{Exp}(\mathcal{U}) = \{[a, A] : a \in \mathcal{AR}, A \in \mathcal{U}\}$ (it is a topology because \mathcal{U} is an ultrafilter). By Theorem 3.12, E is a Baire space and the Baire subsets of E are exactly the \mathcal{U} -Ramsey sets. E is also a ccc space because the partial order $\mathbb{M}_{\mathcal{U}}$ is ccc. The identity function $i : E \rightarrow \mathcal{R}$ is continuous, if we consider \mathcal{R} with the metric separable topology inherited from $\mathcal{AR}^{\mathbb{N}}$. So by Lemma 7.3, every set of reals \mathcal{X} in $L(\mathbb{R})$ is \mathcal{U} -Ramsey. In particular, if such \mathcal{X} is dense in (\mathcal{R}, \leq^*) , then there exists $A \in \mathcal{U}$ such that $[\emptyset, A] \subseteq \mathcal{X}$. Therefore, \mathcal{U} is (\mathcal{R}, \leq^*) -generic over $L(\mathbb{R})$. \square

7.2. \mathcal{H} -Ramseyness of definable sets. From Theorem 7.4 it is easy now to prove the following, which lifts Theorem 4.8 of [8] to the most general context. By definable sets we mean elements of $L(\mathbb{R})$. The proof is very similar to that of Theorem 4.8 of [8] so we will leave the details to the reader.

Theorem 7.5. *If there exists a super compact cardinal and $\mathcal{H} \subseteq \mathcal{R}$ is a semiselective coideal, then all definable subsets of \mathcal{R} are \mathcal{H} -Ramsey.*

Nevertheless, in [3], it was proved that the supercompact cardinal is not needed for the case $\mathcal{R} = \mathbb{N}^{[\infty]}$. Namely:

Theorem 7.6 (Di Prisco, Mijares, Uzcátegui; [3]). *Suppose λ is a weakly compact cardinal. Let $V[G]$ be a generic extension by $\text{Col}(\omega, < \lambda)$ such that $\mathbb{N}^{[\infty]} \in V[G]$. Then, if $\mathcal{H} \subseteq \mathbb{N}^{[\infty]}$ is a semiselective coideal in $V[G]$, every subset of $\mathbb{N}^{[\infty]}$ in $L(\mathbb{R})$ of $V[G]$ is \mathcal{H} -Ramsey.*

Now we will use the results in Section 6.2 to generalize Theorem 7.6 to the context of any topological Ramsey space:

Theorem 7.7. *Let \mathcal{R} be a topological Ramsey space. Suppose λ is a weakly compact cardinal. Let $V[G]$ be a generic extension by $\text{Col}(\omega, < \lambda)$ such that $\mathcal{R} \in V[G]$. Then, if $\mathcal{H} \subseteq \mathcal{R}$ is a semiselective coideal in $V[G]$, every subset of \mathcal{R} in $L(\mathbb{R})$ of $V[G]$ is \mathcal{H} -Ramsey.*

Corollary 7.1. Let \mathcal{R} be a topological Ramsey space. If there is a weakly compact cardinal then for every semiselective coideal $\mathcal{H} \subseteq \mathcal{R}$ all definable subsets of \mathcal{R} are \mathcal{H} -Ramsey.

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